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Probability Densities in Configuration Space and the Hamilton–Jacobi Equation

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The characteristic solutions of the Hamilton–Jacobi equation give the energies of conservative physical systems as functions of position and time. It is shown that these expressions are useful in the formation of probability densities in configuration space for canonical ensembles. Applications are given and discussed.

1. INTRODUCTION

In statistical physics, the most important form of an ensemble of conservative physical systems is the so-called "canonical ensemble" introduced by Gibbs (Tolman, 1967) and defined by means of the Hamiltonian of the considered system, as

$$\rho(q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n) = \rho_0 \exp\left(-\frac{H(q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n)}{\varepsilon}\right)$$

where H is the Hamiltonian, $\varepsilon = \text{const}$, the standard deviation, and ρ_0 is the normalization coefficient. The canonical ensemble so defined gives distributions (more accurately, probability densities) of energies in phase space, which have been very successful in the study of statistical equilibrium.

If we have a relation

$$H(q_1, q_2, \ldots, q_n, p_1 p_2, \ldots, p_n) = E(q_1, q_2, \ldots, q_n, t)$$

between the Hamiltonian function and a function E giving the total energy in terms of position and time, then the theory of probability permits us to

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replace, under certain conditions (Papoulis, 1965), the function H by E in the exponent of the expression of ρ; that is, we can introduce the correspondence

$$\rho(q_i, p_i) \to f(q_i, t) = f_0 \exp\left(-\frac{E(q_i, t)}{\lambda}\right)$$

where now the parameters f_0 and λ may depend on time. If the integral

$$I(t) = \int_V \cdots \int f(q_1, q_2, \ldots, q_n, t)\, dq_1 \cdots dq_n$$

exists in a volume V, then the function f can be considered as a probability density function in V at a given time instant.

In the next sections we shall proceed to the derivation of such functions $f(q_i, t)$ for some standard physical systems. Our computation will be based on solving the Hamilton–Jacobi equation by the method of the characteristics. We shall restrict our study to nonquantal cases.

We note that the idea of deriving probability amplitudes in configuration space was first submitted by Feynman and Hibbs (1965). In this paper we consider the problem from another point of view, using a different, if we are allowed to say, more mathematical, method.

In order to show the utility of the characteristic solutions, let us consider the Hamilton–Jacobi equation that corresponds to a conservative system, namely

$$\frac{\partial S}{\partial t} + H\left(q_1, q_2, \ldots, q_n, \frac{\partial S}{\partial q_1}, \frac{\partial S}{\partial q_1}, \ldots, \frac{\partial S}{\partial q_n}\right) = 0$$

The above equation, solved by the usual method of separation of variables, gives $S = W(q_1, \ldots, q_n) - Et$, and the energy results as $-\partial S/\partial t = E$, where E is an "absolute" constant not depending on space and time. On the other hand, the so-called "characteristic solutions" (John, 1978; Courant and Hilbert, 1962) of the Hamilton–Jacobi equation permit us to find the energies as functions of position and time. Such a solution is the well-known one of the free particles $S = (m/2t)(q - q_0)^2$ with energy $E = m(q - q_0)^2/2t^2$ (Feynman and Hibbs, 1965). This energy yields the probability density of normal form

$$f = \frac{1}{t}\left(\frac{m}{2\pi\varepsilon}\right)^{1/2} \exp\left(-\frac{m(q - q_0)^2}{2t^2\varepsilon}\right)$$

with variance $\sigma^2 = 2t^2\varepsilon/m$.

We shall present several applications which will exhibit the meaning and potential usefulness of the method. We shall see that this study leads to some elegant and interesting results.

2. HARMONIC OSCILLATOR

The Hamilton–Jacobi equation of a linear harmonic oscillator is

$$\frac{\partial S}{\partial t}+\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2+\frac{m\omega^2}{2}q^2=0 \tag{1}$$

with the Hamiltonian

$$H=\frac{P^2}{2m}+\frac{m\omega^2}{2}q^2$$

Following the method of characteristics, we form the characteristic system of equations (1), namely

$$\begin{aligned}&\frac{dq}{dt}=\frac{P}{m}, \qquad \frac{dP}{dt}=-m\omega^2 q\\ &\frac{dH}{dt}=0, \quad \frac{dS}{dt}=\frac{p^2}{2m}-\frac{m\omega^2}{2}q\end{aligned} \tag{2}$$

The general solution of (2) is

$$\begin{aligned}q(t)&=c_1\cos\omega t+c_2\sin\omega t\\ P(t)&=-m\omega(c_1\sin\omega t-c_2\cos\omega t)\\ H&=\frac{m\omega^2}{2}(c_1^2+c_2^2)\\ S(t)&=\frac{m\omega}{4}[(c_2^2-c_1^2)\sin 2\omega t+2c_1c_2\cos 2\omega t]\end{aligned}$$

or we obtain the following two partial solutions:

$$\begin{aligned}q&=c_1\cos\omega t\\ P&=-m\omega c_1\sin\omega t\\ H&=\frac{m\omega^2}{2}c_1^2\\ S&=-\frac{m\omega}{4}c_1^2\sin 2\omega t\end{aligned} \tag{3}$$

and

$$\begin{aligned}q&=c_2\sin\omega t\\ P&=m\omega c_2\cos\omega t\\ H&=\frac{m\omega^2}{2}c_2^2\\ S&=\frac{m\omega}{4}c_2^2\sin 2\omega t\end{aligned} \tag{4}$$

The solutions (3) and (4) form two so-called "characteristic strips" of equation (1). From these by elimination of the constants in the corresponding expressions of S we get two characteristic solutions of (1), namely

$$\left.\begin{aligned} S_{\mathrm{I}} &= -\frac{m\omega}{2} q^2 \tan \omega t \\ S_{\mathrm{II}} &= \frac{m\omega}{2} q^2 \cot \omega t \end{aligned}\right\} \tag{5}$$

Accordingly the energies are

$$\left.\begin{aligned} E_{\mathrm{I}} &= \frac{m\omega^2}{2} \frac{q^2}{\cos^2 \omega t} \\ E_{\mathrm{II}} &= \frac{m\omega^2}{2} \frac{q^2}{\sin^2 \omega t} \end{aligned}\right\} \tag{6}$$

From the above expressions of the energies we form the following probability densities in configuration space

$$\begin{aligned} f_{\mathrm{I}} &= \frac{\omega}{\cos \omega t} \left(\frac{m}{2\pi\varepsilon}\right)^{1/2} \exp\left\{-\frac{m\omega^2}{2\varepsilon} \frac{q^2}{\cos^2 \omega t}\right\} \\ f_{\mathrm{II}} &= \frac{\omega}{\sin \omega t} \left(\frac{m}{2\pi\varepsilon}\right)^{1/2} \exp\left\{-\frac{m\omega^2}{2\varepsilon} \frac{q^2}{\sin^2 \omega t}\right\} \end{aligned} \tag{7}$$

which are normal with periodic time-dependent standard deviations

$$\sigma_1 = \frac{\cos \omega t}{\omega} \left(\frac{\varepsilon}{m}\right)^{1/2} \quad \text{and} \quad \sigma_2 = \frac{\sin \omega t}{\omega} \left(\frac{\varepsilon}{m}\right)^{1/2}$$

The constant ε depends on the particular ensemble considered (Schrödinger, 1967). We note that the functions (7), together with their associated velocities

$$v_{\mathrm{I}} = \frac{1}{m} \frac{\partial S_{\mathrm{I}}}{\partial q} = -\omega q \tan \omega t, \qquad v_{\mathrm{II}} = \frac{1}{m} \frac{\partial S_{\mathrm{II}}}{\partial q} = -\omega q \cot \omega t$$

satisfy the equation

$$v_\alpha \frac{\partial f_\alpha}{\partial q} + f_\alpha \frac{\partial v_\alpha}{\partial q} + \frac{\partial f_\alpha}{\partial t} = 0, \qquad \alpha = \mathrm{I}, \mathrm{II} \tag{8}$$

i.e., the equation of continuity for an ensemble of particles described by the model.

3. ELECTRON IN UNIFORM MAGNETIC FIELD

This case consists of a three-dimensional problem. In order to simplify it, we consider the standard vector potential

$$\mathbf{A} = \tfrac{1}{2}(\mathbf{H} \times \mathbf{r}) = \left(-\frac{Hy}{2}, \frac{Hx}{2}, 0\right)$$

The Hamilton–Jacobi equation then is

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\left[(\nabla S)^2 + \frac{m^2\omega^2}{4}(x^2+y^2) + m\omega\left(y\frac{\partial S}{\partial x} - x\frac{\partial S}{\partial y}\right)\right] = 0 \tag{9}$$

and the Hamiltonian

$$H = \frac{P^2}{2m} + \frac{m\omega^2}{8}(x^2+y^2) + \frac{\omega}{2}(yP_x - xP_y), \qquad \omega = \frac{eH}{mc}$$

The characteristic system is

$$\left.\begin{array}{llll} \dfrac{dx}{dt} = \dfrac{P_x}{m} + \dfrac{\omega}{2}y, & \dfrac{dy}{dt} = \dfrac{P_y}{m} - \dfrac{\omega}{2}x & & \\ \dfrac{dP_x}{dt} = -\dfrac{m\omega^2}{4}x + \dfrac{\omega}{2}P_y, & \dfrac{dP_y}{dt} = -\dfrac{m\omega^2}{4}y - \dfrac{\omega}{2}P_x & & \\ \dfrac{dz}{dt} = \dfrac{P_z}{m}, \quad \dfrac{dP_z}{dt} = 0, & \dfrac{dH}{dt} = 0, & \dfrac{dS}{dt} = \dfrac{P^2}{2m} - \dfrac{m\omega^2}{8}(x^2+y^2) & \end{array}\right\} \tag{10}$$

and has the general solution

$$\left.\begin{array}{l} x = C_1 \sin\omega t + C_2 \cos\omega t + C_3 \\ y = -C_2 \sin\omega t + C_1 \cos\omega t + C_4 \\ P_x = \dfrac{m\omega}{2}(-C_2 \sin\omega t + C_1 \cos\omega t - C_4) \\ P_y = -\dfrac{m\omega}{2}(C_1 \sin\omega t + C_2 \cos\omega t - C_3) \\ z = \dfrac{P_{0z}}{m}t + z_0, \qquad P_z = P_{0z} = \text{const}, \quad H = \text{const} \end{array}\right\} \tag{11}$$

$$S = \frac{m\omega}{2}[(C_1C_3 - C_2C_4)\cos\omega t - (C_2C_3 + C_1C_4)\sin\omega t] + \frac{P_{0z}^2}{2m}t \tag{12}$$

Inspection shows that the characteristic strips result from the following partial solutions:

$$\left.\begin{aligned} x &= C_1 \sin \omega t + C_3 \\ y &= C_1 \cos \omega t \\ P_x &= \frac{m\omega}{2} C_1 \cos \omega t \\ P_y &= -\frac{m\omega}{2} C_1 \sin \omega t \\ S &= \frac{m\omega}{2} C_1 C_3 \cos \omega t + \frac{P_{0z}^2}{2m} t \end{aligned}\right\} \quad (13)$$

$$\left.\begin{aligned} x &= C_1 \sin \omega t \\ y &= C_1 \cos \omega t + C_4 \\ P_x &= \frac{m\omega}{2} (C_1 \cos \omega t - C_4) \\ P_y &= -\frac{m\omega}{2} C_1 \sin \omega t \\ S &= -\frac{m\omega}{2} C_1 C_4 \sin \omega t + \frac{P_{0z}^2}{2m} t \end{aligned}\right\} \quad (14)$$

$$\left.\begin{aligned} x &= C_2 \cos \omega t \\ y &= (-C_2 \sin \omega t - C_4) \\ P_x &= \frac{m\omega}{2} (-C_2 \sin \omega t - C_4) \\ P_y &= -\frac{m\omega}{2} C_2 \cos \omega t \\ S &= -\frac{m\omega}{2} C_2 C_4 \cos \omega t + \frac{P_{0z}^2}{2m} t \end{aligned}\right\} \quad (15)$$

$$\left.\begin{aligned} x &= C_2 \cos \omega t + C_3 \\ y &= -C_2 \sin \omega t \\ P_x &= -\frac{m\omega}{2} C_2 \sin \omega t \\ P_y &= -\frac{m\omega}{2} (C_2 \cos \omega t - C_3) \\ S &= -\frac{m\omega}{2} C_2 C_3 \sin \omega t + \frac{P_{0z}^2}{2m} t \end{aligned}\right\} \quad (16)$$

while in all the above solutions we consider $z = (P_{0z}/m)t + z_0$, $P_z = P_{0z} = \text{const}$, $H = \text{const}$.

From the above four "strips" we get the corresponding solutions

$$\left.\begin{aligned} S_{1a} &= -\frac{m\omega}{2}x^2 \tan \omega t - \frac{m\omega}{2}xy + \frac{m(z-z_0)^2}{2t} \\ S_{1b} &= \frac{m\omega}{2}x^2 \cot \omega t - \frac{m\omega}{2}xy + \frac{m(z-z_0)^2}{2t} \end{aligned}\right\} \tag{17}$$

$$\left.\begin{aligned} S_{2a} &= -\frac{m\omega}{2}y^2 \tan \omega t + \frac{m\omega}{2}xy + \frac{m(z-z_0)^2}{2t} \\ S_{2b} &= \frac{m\omega}{2}y^2 \cot \omega t + \frac{m\omega}{2}xy + \frac{m(z-z_0)^2}{2t} \end{aligned}\right\} \tag{18}$$

The energies $E = -\partial S/\partial t$ result as

$$\left.\begin{aligned} E_{1a} &= \frac{m\omega^2 x^2}{2\cos^2 \omega t} + \frac{m(z-z_0)^2}{2t^2} \\ E_{1b} &= \frac{m\omega^2 x^2}{2\sin^2 \omega t} + \frac{m(z-z_0)^2}{2t^2} \end{aligned}\right\} \tag{19}$$

$$\left.\begin{aligned} E_{2a} &= \frac{m\omega^2 y^2}{2\cos^2 \omega t} + \frac{m(z-z_0)^2}{2t^2} \\ E_{2b} &= \frac{m\omega^2 y^2}{2\sin^2 \omega t} + \frac{m(z-z_0)^2}{2t^2} \end{aligned}\right\} \tag{20}$$

So, for the case of electrons in a uniform magnetic field we distinguish the following four probability density functions:

$$\left.\begin{aligned} f_{1a} &= \frac{m\omega}{2\pi\varepsilon t \cos \omega t} \exp\left\{-\frac{m}{2\varepsilon}\left(\frac{\omega^2}{\cos^2 \omega t}x^2 + \frac{(z-z_0)^2}{t^2}\right)\right\} \\ &= f_\perp(x;t)\varphi_\parallel(z;t) \\ f_{1b} &= \frac{m\omega}{2\pi\varepsilon t \cos \omega t} \exp\left\{-\frac{m}{2\varepsilon}\left(\frac{\omega^2 x}{\sin^2 \omega t}x^2 + \frac{(z-z_0)^2}{t^2}\right)\right\} \\ &= g_\perp(x;t)\varphi_\parallel(z;t) \end{aligned}\right\} \tag{21}$$

$$
\left.\begin{aligned}
f_{2a} &= \frac{m\omega}{2\pi\varepsilon t\cos\omega t}\exp\left\{-\frac{m}{2\varepsilon}\left(\frac{\omega^2}{\cos^2\omega t}y^2+\frac{(z-z_0)^2}{t^2}\right)\right\}\\
&= f_\perp(y;t)\varphi_\parallel(z;t)\\
f_{2b} &= \frac{m\omega}{2\pi\varepsilon t\sin\omega t}\exp\left\{-\frac{m}{2\varepsilon}\left(\frac{\omega^2}{\sin^2\omega t}y^2+\frac{(z-z_0)^2}{t^2}\right)\right\}\\
&= g_\perp(y;t)\varphi_\parallel(z;t)
\end{aligned}\right\} \tag{22}
$$

Comparing the results (19)–(22) with those of the previous sections, we see that the model shows the electrons to behave like free particles in the direction parallel to the field and like linear harmonic oscillators in the plane perpendicular to the field. In particular, the densities show that the probability of finding a particle at some given position on the level perpendicular to the field and that of finding a particle along the direction of the field are statistically independent. These results are plausible and consistent with the theory of electrodynamics and observations. But let us now investigate what happens with the gauge invariance.

4. GAUGES OF THE VECTOR POTENTIAL

With regard to equation (9) let us now study the following two Hamilton–Jacobi equations:

$$
\frac{\partial S}{\partial t}+\frac{1}{2m}(\nabla S)^2+\frac{m\omega^2}{2}y^2+\omega y\frac{\partial S}{\partial x}=0 \tag{23}
$$

$$
\frac{\partial S}{\partial t}+\frac{1}{2m}(\nabla S)^2+\frac{m\omega^2}{2}x^2-\omega x\frac{\partial S}{\partial y}=0 \tag{24}
$$

resulting respectively, for the gauges of the vector potential $\mathbf{A}_1=(-Hy,0,0)$, $\mathbf{A}_2=(0,Hx,0)$, and see whether the gauge invariance is preserved in the present theory.

Using the same method as previously, we find the following solutions for equation (23):

$$
\left.\begin{aligned}
S_{\mathrm{Ia}} &= -\frac{m\omega}{2}x^2\tan\omega t-m\omega xy+\frac{m(z-z_0)^2}{2t}\\
S_{\mathrm{Ib}} &= \frac{m\omega}{2}x^2\cot\omega t-m\omega xy+\frac{m(z-z_0)^2}{2t}
\end{aligned}\right\} \tag{25}
$$

with the energies

$$\left.\begin{aligned} E_{\mathrm{Ia}} &= \frac{m\omega^2 x^2}{2\cos^2\omega t} + \frac{m(z-z_0)^2}{2t^2} \\ E_{\mathrm{Ib}} &= \frac{m\omega^2 x^2}{2\sin^2\omega t} + \frac{m(z-z_0)^2}{2t^2} \end{aligned}\right\} \tag{26}$$

The probability densities for the case of the gauge potential $(-Hy, 0, 0)$ are

$$\left.\begin{aligned} f_{\mathrm{Ia}} &= \frac{m\omega}{2\pi\varepsilon t\cos\omega t}\exp\left\{-\frac{m}{2\varepsilon}\left(\frac{\omega^2}{\cos^2\omega t}x^2 + \frac{(z-z_0)^2}{t^2}\right)\right\} \\ &= f_\perp(x;t)\varphi_\parallel(z;t) \\ f_{\mathrm{Ib}} &= \frac{m\omega}{2\pi\varepsilon t\sin\omega t}\exp\left\{-\frac{m}{2\varepsilon}\left(\frac{\omega^2}{\sin^2\omega t}x^2 + \frac{(z-z_0)^2}{t^2}\right)\right\} \\ &= g_\perp(x;t)\varphi_\parallel(z;t) \end{aligned}\right\} \tag{27}$$

We note that the above functions coincide exactly with (21) of the previous section.

With regard to equation (24), corresponding to the gauge $(0, Hx, 0)$, we have the solutions

$$\left.\begin{aligned} S_{\mathrm{IIa}} &= -\frac{m\omega}{2}y^2\tan\omega t + m\omega xy + \frac{m(z-z_0)^2}{2t} \\ S_{\mathrm{IIb}} &= \frac{m\omega}{2}y^2\cot\omega t + m\omega xy + \frac{m(z-z_0)^2}{2t} \end{aligned}\right\} \tag{28}$$

with the energies

$$\left.\begin{aligned} E_{\mathrm{IIa}} &= \frac{m\omega^2 y^2}{2\cos^2\omega t} + \frac{m(z-z_0)^2}{2t} \\ E_{\mathrm{IIb}} &= \frac{m\omega^2 y^2}{2\sin^2\omega t} + \frac{m(z-z_2)^2}{2t} \end{aligned}\right\} \tag{29}$$

and the respective probability densities

$$\left.\begin{aligned} f_{\mathrm{IIa}} &= \frac{m\omega}{2\pi\varepsilon t\cos\omega t}\exp\left\{-\frac{m}{2\varepsilon}\left(\frac{\omega^2}{\cos^2\omega t}y^2 + \frac{(z-z_0)^2}{t^2}\right)\right\} \\ &= f_\perp(y;t)\varphi_\parallel(z;t) \\ f_{\mathrm{IIb}} &= \frac{m\omega}{2\pi\varepsilon t\sin\omega t}\exp\left\{-\frac{m}{2\varepsilon}\left(\frac{\omega^2}{\sin^2\omega t}y^2 + \frac{(z-z_0)^2}{t^2}\right)\right\} \\ &= g_\perp(y;y)\varphi_\parallel(z;t) \end{aligned}\right\}. \tag{30}$$

Again the above functions coincide with the ones given by equations (22). So we conclude that gauge invariance is preserved with regard to the probability densities, but not absolutely. We have seen that the "complete" Hamilton–Jacobi equation (9) contains the densities of both the gauge-transformed ones (23) and (24). Also, we note that the results (25) and (28) are not identical, and in order to make them coincide we have to apply the orthogonal transformation in x, y

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

i.e., a $\pi/2$ rotation of the xy plane.

The content of this and the previous section suggests two slightly different behaviors of electrons in a uniform magnetic field. It is possible that this fact has some connection with the spin property, where also one observes a splitting in the behavior of two otherwise identical electrons. But let us leave this idea for a future study.

Finally, we remark that one can easily verify that the velocities $\mathfrak{v} = (1/m)$ grad S and their associated probability densities satisfy the equation of continuity

$$\mathfrak{v} \operatorname{grad} f + f \operatorname{div} \mathfrak{v} + \frac{\partial f}{\partial t} = 0$$

5. ATTRACTIVE POTENTIAL

A standard two-dimensional problem is the problem of an attractive potential. In this case we consider the Hamilton–Jacobi equation

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\left[\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial \vartheta}\right)^2\right] - \frac{\alpha^2}{r} = 0 \tag{31}$$

and the Hamiltonian

$$H = \frac{1}{2m}\left(P_r^2 + \frac{1}{r^2} P_\vartheta^2\right) - \frac{\alpha^2}{r}, \qquad \alpha = \text{const} \tag{32}$$

The characteristic system is

$$\begin{aligned} &\frac{dr}{dt} = \frac{P_r}{m}, \qquad \frac{d\vartheta}{dt} = \frac{P_\vartheta}{mr^2} \\ &\frac{dP_r}{dt} = \frac{P_\vartheta^2}{mr^3} - \frac{\alpha^2}{r^2}, \qquad \frac{dP_\vartheta}{dt} = 0 \\ &\frac{dH}{dt} = 0, \qquad \frac{dS}{dt} = H + \frac{2\alpha^2}{r} \end{aligned} \tag{33}$$

from which eventually we have $H = H_c = \text{const}$ and $P_\vartheta = P_{\vartheta c} = \text{const}$. Introducing the functions

$$R(r) = 2mH_c r^2 + 2m\alpha^2 r - P_{\vartheta c}^2 \tag{34}$$

$$I(r) = \int \frac{dr}{[R(r)]^{1/2}} \tag{35}$$

and the relation (Landau and Lifchitz, 1966)

$$mr\frac{dr}{\sqrt{R}} = dt \tag{36}$$

we get the solution of the canonical system in the implicit form

$$\sqrt{R} - m\alpha^2 I(r) = 2H_c(t + t_c) \tag{37}$$

$$\sin(\vartheta + \vartheta_c) = \frac{m\alpha^2 r - P_{\vartheta c}^2}{r(2mH_c P_{\vartheta c}^2 + m^2\alpha^4)^{1/2}} \tag{38}$$

The equation in $S(t)$ gives, by means of equations (36) and (37), the solution

$$S = 2\sqrt{R} - 3H_c t \tag{39}$$

The constant H_c can be evaluated from the relation

$$\frac{dH_c}{dt} = \frac{\sqrt{R}}{mr}\frac{dH_c}{dr} = 0, \qquad \text{or} \qquad \frac{dH_c}{dr} = 0$$

From (37) we have

$$H_c = \frac{1}{2(t + t_c)}\left(\sqrt{R} - m\alpha^2 \int \frac{dr}{\sqrt{R}}\right)$$

The above relation gives

$$H_c = 0 \tag{40}$$

Consequently the solution (39) becomes

$$S = 2(2m\alpha^2 r - P_{\vartheta c}^2)^{1/2} \tag{41}$$

The constant $P_{\vartheta c}$ is expressed in terms of r, ϑ by means of equation (38) for $H_c = 0$, as

$$P_{\vartheta c}^2 = m\alpha^2 r[1 - \sin(\vartheta + \vartheta_c)] \tag{42}$$

By means of (42) and (41) we finally get the characteristic solution of our initial equation (31), namely

$$S(r, \vartheta) = 2\{m\alpha^2 r[1 - \sin(\vartheta + \vartheta_c)]\}^{1/2} \tag{43}$$

The resulting energy is $-\partial S/\partial t = E = 0$.

We see that the characteristic energy in the present case vanishes. Let us call such a physical system [i.e., with $E = -\partial S(q, t)/\partial t = 0$] a "hyperconservative system."

The characteristic density here is expected to be represented by a nonnegative function of the form $f = f_0 u(r, \vartheta)$ and such that $\int_0^\infty \int_0^{2\pi} f(r, \vartheta) r\, dr\, d\vartheta < \infty$.

The property $\partial f/\partial t = 0$ means that there is no spreading of the distribution of hyperconservative systems with time. This fact is significant, since it would explain the observed stability and very long lifetime of certain physical systems of aggregates of particles or bodies.

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